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A DEGENERATE HARTMAN THEOREM

BY

THOMAS J. MORRISEY

Summit Systems
20 Exchange Place
New York, NY 10005, USA

ABSTRACT

Consider a real analytic diffeomorphism, $f\colon \mathbf{R}^2 \to \mathbf{R}^2$, with q as a non-hyperbolic fixed point and $Df(q) = \mathrm{Id}$. Placing sufficient conditions on lowest-order non-linear terms in the expansion of f, we show the function is topologically conjugate with a decoupled product map. The impetus for studying such a function arose in the classical three-body problem.

1. Introduction

Imposing non-trivial sufficient conditions on a 2-dimensional diffeomorphism ensures the diffeomorphism is topologically conjugate to a decoupled map. This conjugacy is in a quasi-sector of a non-hyperbolic equilibrium point, hence the title.

Such diffeomorphisms are Poincaré maps in several applications. For example, these conditions are more general than those arising naturally in several settings of the well-known three-body problem, [1].

Consider a vector bundle with base space a 2-manifold, having fiber dimension 2. A vector cone is formed by first fixing an element of the base space, then choosing a particular subset within the fiber over this element. This particular subset is closed under addition and scalar multiplication. An orbit for an element of the base space and its associated cone field are studied. To this end ponder any transversal along the stable manifold of the equilibrium point, along with its tangent space.

As a transversal iterated by the diffeomorphism, the tangent space is acted on by the derivative of this diffeomorphism. Observing how this derivative acts on a

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vector cone reveals that properly chosen transversals will not "wiggle" too much. A foliation of dimension one of the base space is then defined, the iterates of a properly chosen transversal the leaves.

Applying the above ideas, *mutatis mutandis*, to the inverse of the diffeomorphism, yields a coordinate system with domain the base space. Construction of the topological equivalence then follows from geometric methods, [2].

2. Statement of results

Let U be a manifold and $\Phi\colon U \to \Phi(U)$ be a diffeomorphism with a fixed point $q \in U$. That is, $\Phi(U)$ has non-empty intersection with U. Define the **stable set of** q to be points in U that are **forward asymptotic to** q,

$$W^S(q, \Psi) = \{y \in U\colon \Phi^k(y) \to q \quad \text{as } k \to \infty\}.$$

Define the **unstable set of** q to be points in U, **backward asymptotic to** q,

$$W^U(q, \Phi) = \{y \in U\colon \Phi^k(y) \to q \quad \text{as } k \to -\infty\}.$$

When the context is clear, write W^S and W^U respectively.

A function $\phi \in O(N+1)$, $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$, if for (x, y) near the origin there exists some constant $C \in \mathbb{R}$ such that $\|\phi(x,y)\| \leq C\|(x,y)\|^{N+1}$. Write ϕ_x, ϕ_y to denote respective partial derivatives of a given function ϕ. Further, write $\phi \in C^\infty$ if ϕ has continuous partial derivatives of all orders. The letter t used in superscripting indicates transpose.

Analysis for our map, f, takes place in a subset, U, of the first quadrant $Q1 = \{(x,y) \in \mathbb{R}^2\colon x \geq 0 \text{ and } y \geq 0\}$. This subset U is dependent on the map f, and is always sufficiently small. And now the main theorem.

THEOREM: *Let $f\colon U \to f(U)$ be a diffeomorphism of the form*

$$f(x,y) = \begin{pmatrix} x(1 - p(x,y) - r\ (x,y)) \\ y(1 + p(x,y) + t\ (x,y)) \end{pmatrix}^t \tag{2.1}$$

with the right-hand side having higher-order terms described by

$$p(x,y) = \sum_{j=0}^{n} a_j x^j y^{N-j}, \quad a_j \in \mathbb{R}, \quad \forall j; \quad r, t \in C^\infty \cap O(N+1).$$

Assume $p > 0$, $p_x > 0$, $p_y > 0$ and $p_x p_y > p p_{xy}$, in $Q1 - \{(0,0)\}$. Then the diffeomorphism f is topologically conjugate to the decoupled product map

$$f|W^S \times f|W^U.$$

Observe that the map, f, is a local diffeomorphism at the origin. Also $(0,0)$ is a non-hyperbolic stationary point for f. This with the conditions in the main theorem imply those in [1, Theorem 1] for both f and f^{-1}. Therefore f and f^{-1} have stable sets of the origin given by graphs of differentiable functions. In fact the graphs are one-dimensional C^∞ manifolds, [3].

Any diffeomorphism satisfying the hypothesis of the main theorem will be referred to as **of the form (2.1)**. Placing the condition $p_x p_y > p p_{xy}$ on the homogeneous polynomial p is redundant if p has positive coefficients; this is shown in section 4.

3. Remarks

Remarks in this section establish some generality of the setting. To begin, an angular-valued function $\theta = \theta(r)$, $r \in [0, r_0]$, is a **μ-radial curve** if θ is a Lipschitz function of the radial variable r with Lipschitz constant μ. A **μ-sector** is formed by two μ-radial curves that intersect only at $r = 0$. These definitions are similar to the definitions for μ-vertical curves (μ-horizontal curves), and μ-vertical strips (μ-horizontal strips) given in [4].

Consider another diffeomorphism $g\colon Q \to g(Q)$; Q is the μ-sector formed by the μ-radial curves θ_1 and θ_2 emanating from the origin. For this section's purposes, and without loss of generality, assume that g has $(0,0)$ as a fixed point. These curves will represent the stable set of the origin and unstable set of the origin for g. Ideally this map, g, is, up to an analytic change of coordinates, of the form (2.1).

Assume the curves θ_1 and θ_2 are differentiable functions or r that intersect transversally at the origin, write $\Theta_1(0)$ for $D_r\theta_1(0)$, and write $\Theta_2(0)$ for $D_r\theta_2(0)$.

Suppose that the contractive rate of movement along θ_1 is non-linear and the expansive rate along θ_2 is non-linear. Assume the lowest-order non-linear terms that give the contraction rate and the lowest-order non-linear terms that give the expansive rate are of the same order.

Let $[[r, \theta]]$ be the polar representative of (x, y), so $r^2 = x^2 + y^2$ and $\theta =$

$\tan^{-1}(y/x)$. Define the function

$$G((x,y)) = G([[r,\theta]]) = [[r,(\pi/2)(\theta - \Theta_1(0)/(\Theta_2(0) - \Theta_1(0))]].$$

The composite function $G \circ g \circ G^{-1}$ positions Q into a neighborhood of Q1, such that the stable set of the origin is tangent to the x-axis at the origin. Also, the unstable set of the origin is positioned tangent to the y-axis at the origin. Movement defined by g in Q can be studied via movement defined by $G \circ g \circ G^{-1}$ in Q1, since $G \circ g \circ G^{-1}$ would be in the form given in [1, Theorem 1].

Write W_g^S for the **center**-stable manifold of the origin, for g, and write W_g^U for the **center**-unstable manifold of the origin, for g. Then near the origin, $W_g^S = \{(x,\omega_1(x))\}$ and $W_g^U = \{(\omega_2(y),y)\}$, where the ω_i are C^∞ functions. Another C^∞ coordinate change, $\Omega(x,y) = (x - \omega_2(y), y - \omega_1(x))$, positions W_g^S locally onto the x-axis and places W_g^U locally onto the y-axis. Summarily, these changes of coordinate systems hopefully produce a diffeomorphism of the form (2.1).

4. Lemma

For a given map $f\colon U \to f(U)$, and given an element (x,y) in U, the **orbit of** (x,y) **in** U, or simply **orbit** when the context is clear, is defined as

$$\mathrm{Or}(x,y) = \{f^k(x,y) \in U\colon k \in Z\}.$$

Interest in orbits of sections within the base space is fundamental, therefore notation involving evaluation of functions along an orbit is numerous. To simplify, write (x_k,y_k) for the kth iterate of (x,y) by f, that is, $f^k(x,y) = (x_k,y_k)$. Additionally write $p_k = p(x_k,y_k)$, etc., where the notation for a function evaluated at (x_k,y_k) is simplified by way of subscripting when no previous subscript exists. If the function already has a subscript, as in a partial derivative, then superscripting is used.

In the supportive arguments as well as the preceding comments, concepts about **cones** are key. Here cones are obtained by taking a vector bundle and fixing an element of the base space, then choosing a subset of the fiber over this element. This subset is closed under addition and scalar multiplication.

For the following purposes, define a **vector cone** V, with **basepoint** (x,y), and **endpoints** a and b:

$$V = V((x,y),a,b) = \{((x,y),\alpha\binom{v}{1})\colon a \le v \le b, \alpha \in \mathbf{R}\}.$$

Endpoints may depend on the basepoint, $a = a(x, y)$ and $b = b(x, y)$.

Given any indexing set K, a **cone field**, $V_K = (V((x_k, y_k), a_k, b_k))_{k\in K}$, is a family of vector cones. An **invariant cone field for** f is a cone field indexed by some subset H, of the integers, Z, such that

$$Df(x_k, y_k)[V((x_k, y_k), a_k, b_k)] \subset V((x_{k+1}, y_{k+1}), a_{k+1}, b_{k+1}), \quad \forall k \in H.$$

Two cone fields are **disjoint** if they are disjoint as sets. That is, the cone fields V_K and V_M are disjoint if

$$\left(\bigcup_{k\in K} V((x_k, y_k), a_k, b_k)\right) \cap \left(\bigcup_{n\in M} V((x_n, y_n), a_n, b_n)\right) = \{\quad\}.$$

A real-valued **inclination** function, λ, is called into action on the vector bundle $U \times \mathbb{R}^2$, as in [2]. This facilitates description of movement of a transversal inside the base space. To inaugurate this function, fix an element (x, y) in the base space U. Define λ: $\mathbb{R}^2 \to \mathbb{R}$ for any $v = (v_1, v_2)^t$ in the fiber over (x, y),

$$\lambda(x) = \frac{v_1}{v_2}.$$

If $\lambda(v)$ takes a value near zero, then v is nearly vertical. If the value of $\lambda(v)$ is arbitrarily large in magnitude, then v is arbitrarily horizontal.

Notation involving (x, y) in vector cones becomes cumbersome, furthermore the context is generally apparent, hence this portion of the notation will be omitted whenever possible.

Proving the topological conjugacy requires the following lemmas. Some results that these lemmas assert are that points exodus U under iteration by f, and that there exist disjoint invariant cone fields for f and f^{-1}.

LEMMA A: *For any $(x, y) \in U$, $x > 0$ and $y > 0$, U sufficiently small, and f of the form* (2.1), *there exists an iterate (x_k, y_k) which is not an element of U.*

Proof: Initially, U is chosen so that the terms of highest order in the Taylor expansion are dominated by the homogeneous polynomial p.

Given the point $(x, y) \in U$, where $x > 0$ and $y > 0$, construct two line segments, a vertical line segment $LV = LV(x, y) = \{(x, z) \in U\colon z \in [0, y]\}$, and a horizontal line segment $LH = LH(x, y) = \{(w, y) \in U\colon w \in [0, x]\}$.

Since f is of the form (2.1), $p_x > 0$ and $p_y > 0$. Let π_i be the projection map onto the ith component. For $(x, z) \in LV$, the inequality $\pi_1 \circ f(x, y) < \pi_1 \circ f(x, z)$

holds. Also for $(w, y) \in LH$, $\pi_2 \circ f(x,y) > \pi_2 \circ f(w,y)$. In particular, $\pi_1 \circ f(x,y) < \pi_1 \circ f(x,0)$ and $\pi_2 \circ f(x,y) > \pi_2 \circ f(0,y)$, so that $\pi_1 \circ (x_k, y_k) < \pi_1 \circ f^k(x,0)$ and $\pi_2 \circ (x_k, y_k) > \pi_2 \circ f^k(0,y)$, for all $k > 0$. ∎

LEMMA B: *Let q be any homogeneous polynomial of degree $k \geq 0$; then $kq = xq_x + yq_y$.*

LEMMA C: *Let q be any homogeneous polynomial of degree $k \geq 0$ and let f be of the form (2.1); then $q_1 = q - xpq_x + ypq_y + O(2k+1)$.*

Proof: Taylor expansion. ∎

LEMMA D: *Let q be any homogeneous polynomial of degree $k \geq 0$, with $q > 0$ in the sector $S = \{(x,y)\colon r > 0, \tan^{-1}(y/x) \in [\theta_1, \theta_2]\}$. Let $\tau \in O(k+1)$; then there exists some $r_0 > 0$ such that, in the set $S \cap \{0 < \|(x,y)\| \leq r_0\}$, the map $q + \tau$ is strictly positive.*

Proof: Denote the polar representative of (x,y) by $[[r,\theta]]$. Write $q + \tau = r^k(\eta(\theta) + rT(r,\theta))$, where $q = r^k\eta(\theta)$ and $\tau = r^{k+1}T(r,\theta)$ in the polar coordinate system. The assumption that $q > 0$ in S implies that when θ is in the interval $[\theta_1, \theta_2]$, then there exists β such that $\eta(\theta) \geq \beta > 0$. Let M be a constant such that for $r \leq r_M$, with r_M small enough, $\|\tau(x,y)\| \leq M\ r^{k+1}$. Then for $f < \min\{r_M, \beta/2M\}$, and in S,

$$q + \tau \geq q - |\tau| = r^k(\eta(\theta) - r|T(r,\theta)|) \geq r^k(\beta - rM) > r^k(\beta/2) > 0. \quad ∎$$

LEMMA E: *Let f be of the form (2.1), and let $(x_0, y_0) \in U$; then there exist invariant cone fields for f and f^{-1}, defined by $\mathrm{Or}(x_0, y_0)$, which are disjoint.*

Proof: All analysis is in the sufficiently small sector U, which will shrink. In proving this result, a properly chosen cone field is shown to be an invariant cone field. With this intention, take an element (x_0, y_0) in U, with its associated orbit, $\mathrm{Or}(x_0, y_0)$, and define the cone field $V_K = V((x_k, y_k), a_k, b_k)_{k \in K}$, by setting

$$a_k = \frac{-p_y^k}{p_x^k}, \quad b_k = 0.$$

V_K will be shown to be an invariant cone field for f. For this task, make pairwise comparisons of consecutive elements of the orbit. Subsequently, the k used in subscripts is removed, and it will be enough to show that, for all (x, y) and (x_1, y_1) in $\mathrm{Or}(x_0, y_0)$,

$$Df(x,y)[V((x,y),a,b)] \subset V((x_1,y_1),a_1,b_1).$$

Explicitly writing the derivative, Df,

$$Df(x,y) = \begin{pmatrix} 1 - p - xp_x + O(N) & -xp_y + O(N) \\ yp_x + O(N) & 1 + p + yp_y + O(N) \end{pmatrix},$$

note that $p_y > 0$, thus $\lambda(Df(x,y)(0,1)^t) < \lambda((0,1)^t)$. Since the diffeomorphism f is orientation preserving, it remains to show that $\lambda(Df(x,y)(a,1)^t) > \lambda((a_1,1)^t)$.

Recall the condition $pp_{xy} < p_x p_y$ in Q1 $-\{(0,0)\}$. Since $p > 0$ in this set, this is the same as, in Q1 $-\{(0,0)\}$, $2(N-1)ppp_{xy} < 2(N-1)pp_x p_y$. Appealing to Lemma B, in Q1 $-\{(0,0)\}$ this condition becomes

$$(xp_x + yp_y)pp_{xy} + (xp_x + yp_y)pp_{xy} < 2pp_xp_y + 2(N-2)pp_xp_y. \tag{4.1}$$

Rearranging terms yields another equivalent inequality

$$(xp_x + yp_y)pp_{xy} < 2pp_xp_y + ((N-2)p_x - yp_{xy})pp_y + ((N-2)p_y - xp_{xy})pp_x. \tag{4.2}$$

Expanding the left-hand side of (4.2), rearranging terms and applying Lemma B again, produces

$$xpp_xp_{xy} + ypp_yp_{xy} < xpp_yp_{xx} + pp_xp_y + ypp_xp_{yy} + pp_xp_y. \tag{4.3}$$

Adding p_xp_y to both sides of (4.3), and after manipulating terms, (4.3) is seen to be equivalent to

$$p_xp_y - pp_xp_y - xpp_yp_{xx} + ypp_yp_{xy} < p_xp_y + pp_xp_y - xpp_xp_{xy} + ypp_xp_{yy}. \tag{4.4}$$

Making use of Lemma D, with terms of order $2N-2$ fixed, and considering (4.4), the following inequality is valid in $U - \{(0,0)\}$:

$$\begin{aligned} &(p_y - pp_y + O(2N-2))(p_x - xpp_{xx} + ypp_{xy} + O(2N-2)) \\ &\quad < (p_y - xpp_{yx} + ypp_{yy} + O(2N-2))(p_x + pp_x + O(2N-2)). \end{aligned} \tag{4.5}$$

Terms of higher order are the remainder terms from Lemma C and appropriate higher-order terms confronted in application of Df. Roughly speaking, (4.5) is inequality (4.4) with both sides factored.

Use the fact that $p_x > 0$, and $p_y > 0$, then rewrite (4.5) to obtain

$$\frac{p_y(1-p)+O(2N-2)}{p_x(1+p)+O(2N-2)} < \frac{p_y - xpp_{yx} + ypp_{yy} + O(2N-2)}{p_x - xpp_{xx} + ypp_{xy} + O(2N-2)}. \tag{4.6}$$

Observe that the right-hand side of inequality (4.6) is (p_y^1/p_x^1). Add and subtract the quantity xp_xp_y to the numerator of the left-hand side. Also add and subtract the quantity yp_xp_y to the denominator of the left-hand side. In a small U,

$$\lambda(Df(x,y)(-p_y,p_x)^t) > \lambda((-p_y,p_x)^t),$$

showing that V_K is an invariant cone field.

Applying an analogous argument to f^{-1}, the invariant cone field is W_K, comprised of vector cones V, with elements whose fiber coordinate of interest is of the form $(1,v)^t$. Here let the vector cones constructed for the inverse have endpoints $a = -p_x/p_y$ and $b = 0$. The above argument holds when applied *mutatis mutandis* to this cone field, showing that indeed it is invariant. Moreover, this cone field is disjoint from the cone field constructed above for f. ∎

LEMMA F: *Let q be a homogeneous polynomial of degree $k \geq 1$, with non-negative coefficients and at least two positive coefficients; then in Q1, $q_xq_y - qq_{xy} > 0$.*

Proof: Write the polynomial q as

$$q(x,y) = \sum_{i=1}^{k} a_i x^i y^{k-i}.$$

Using the symmetry of the coefficients and the hypothesis on the coefficients, the following inequality holds:

$$\begin{aligned} q_xq_y - qq_{xy} &= \sum_{i=1}^{k-1}\sum_{j=i+1}^{k-1}(j-i)^2 a_i a_j x^{i+j-1} y^{2k-(i+j)-1} \\ &\quad + a_k \sum_{j=1}^{k-1}(k-j)^2 a_j x^{k+j-1} y^{k-j-1} + a_0 \sum_{i=1}^{k} i^2 a_i x^{i-1} y^{2k-i-1} \\ &> 0. \quad ∎ \end{aligned}$$

5. Proof of the Main Theorem

Proof: Take any point $(x_0, 0)$ along the stable manifold, W^S, in a U satisfying the finite number of sufficiently small requirements described above. The interval $[\pi_1(f(x_0,0)), x_0]$ lies along the stable manifold, and $\bigcup_{k\geq 0} f^k([\pi_1(f(x_0,0)), x_0])$ contains W^S, excluding the origin.

Look at a vertical line segment, VL, from $(x_0, 0)$ to (x_0, y_0), where $y_0 \leq x^N$. Consider the image of the vertical line segment, $f(VL)$, and its intersection point, $f(x_0, 0)$, with the stable manifold. Let Γ: $[0,1] \to [0,1]$ be any C^∞ perturbation function with $\Gamma(0) = 0$, $\Gamma(1) = 1$, $\Gamma(t) > 0$ for t in $[0,1]$, and $\Gamma^{(k)}(0) = 0$, for all $k \geq 1$, where the k superscripting is the kth derivative. Define a homotopy γ: $VL \times [0,1] \to f(VL)$ by

$$\gamma((x_0, y), t) = (\Gamma(t) f(x_0, y) + (1 - \Gamma(t))(x_0, y)).$$

Let F be the closed region bounded below by the closed interval $[\pi_1(f(x_0, 0)), x_0]$, bounded above by the curve $\gamma((x_0, y_0), [0,1])$, and on the sides bounded by the vertical line VL and VL's image $f(VL)$. The above conditions are sufficient to show that γ is one-to-one and onto the fundamental domain F.

Consider the horizontal line segment, HL, contained in U, with one end on the unstable manifold and the other end some iterate, m, of (x_0, y_0). From the proof of Lemma E, and the definition of the homotopy γ, the set $\bigcup_{k\geq 0}(f^k(F) \cap U)$ fills the region above W^S, and under the continuous curve, $JC = HL \cup (\bigcup_0^m f^k(\gamma((x_0, y_0), [0,1])))$.

Final modification of the set U, in addition to satisfying the sufficiently small conditions above, is to let U be the region bounded above by the continuous curve JC, on the left by the unstable manifold of the origin W^U, and on the right by the vertical line VL, and beneath by the stable manifold of the origin W^S.

Remark that iterates of the line segment VL and iterates of $\gamma(VL, t)$ define a foliation, $\mathcal{F}_1$, for U. Every point in U is in some unique iterate of a leaf in F. Inverse iterates of HL may be used to define a second foliation, $\mathcal{F}_2$, of U. In defining $\mathcal{F}_2$, the fundamental domain, F', will be the closed region bounded above by HL, below by $f^{-1}(HL)$, on the left by the unstable manifold W^U, and on the right by $f^{m-1}(\gamma((x_0, y_0), [0,1]))$.

To define the foliation of F', take Γ as above, and define γ_1: $HL \times [0,1] \to F'$

by

$$\gamma_1((x,y_m),t) = (\Gamma(1-t)(x,y_m) + (1-\Gamma(1-t))f^{-1}(x,y_m)).$$

The foliation $\mathcal{F}_2$ has leaves defined in F' by $\gamma_1(HL,t)$, and leaves defined in U by $f^{-k}(\gamma(HL,t))$, $k \geq 0$. Therefore every point in U is in some unique iterate of a leaf in F'. For a leaf taken from $\mathcal{F}_1$, Lemma E permits at most one intersection point with a leaf taken from $\mathcal{F}_2$.

Apply this same technology involving Γ and γ to any decoupled map d: $\mathrm{Q1}' \to d(\mathrm{Q1}')$, $d(x,y) = (x - d_1(x), y + d_2(y))$, where the d_i are $O(2)$ and positive. $\mathrm{Q1}'$ is a subset of Q1 with the vertical line segment VL' determining the horizontal line segment, HL', by taking as an endpoint for HL' the mth iterate of the endpoint of VL'.

Since the sequence, $((x_k - d_1(x_k))_{k\geq 0}$, determined by a small initial value of x, is decreasing and bounded below, it converges. Similarly, $(y_k - d_2(y_k))_{k\geq 0}$ converges. In fact, both sequences converge to 0.

Denote this second set of foliations defined with respect to d_1 and d_2, as $\mathcal{P}_1$ and $\mathcal{P}_2$, with leaves of respective fundamental domains denoted by $\gamma_2(VL',t)$ and $\gamma_3(HL',t)$.

Proving the main theorem is the choice of letting d_1 be f restricted to the stable manifold, and letting d_2 be the restriction of f to the unstable manifold.

Points in the sets U and $\mathrm{Q1}'$ can be represented with respect to the coordinate systems that have been defined. With respect to $\mathcal{F}_1$ and $\mathcal{F}_2$, and for (x,y) in U, $(x,y) = f^k(\gamma(VL,t)) \cap f^{-i}(\gamma_1(HL,s))$, for some $s,t \in [0,1]$ and $k,i \in Z$. This slightly maligns the notation. With respect to $\mathcal{P}_1$ and $\mathcal{P}_2$ for (x',y') in $\mathrm{Q1}'$, $(x',y') = d^j(\gamma(VL',t_1)) \cap d^{-h}(\gamma_1(HL',s_1))$, for some $s_1,t_1 \in [0,1]$ and $j,h \in Z$. These representations will be used to construct the topological conjugacy. Define a function $H\colon U \to \mathrm{Q1}'$ by

$$\begin{aligned} H(x,y) &= H(f^k(\gamma((x_0,0),t)) \cap f^{-i}(\gamma_1((0,y_m),s))) \\ &= d^k(\gamma_2(((x_0)',0),t)) \cap d^{-i}(\gamma_3((0,(y_m)'),s)). \end{aligned}$$

That is, H is defined by projection along leaves onto the stable manifold and unstable manifold of the origin. Here $(x_0)'$ determines VL', and $(y_0)'$ determines HL'.

Now to show that $H \circ f = d \circ H$. Let $(x,y) = f^k(\gamma(VL,t)) \cap f^{-i}(\gamma_1(HL,s))$; then for some $(x_0,z) \in VL$, $(w,y_m) \in HL$, $s \in [0,1]$, $t \in [0,1]$, and $k,i \in Z$, write $(x,y) = f^k(\gamma((x_0,z),t)) \cap f^{-i}(\gamma_1((w,y_m),s))$. Observe,

$$\begin{aligned} H \circ f(x,y) &= H(f(f^k(\gamma((x_0,z),t)) \cap f^{-i}(\gamma_1((w,y_m),s)))) \\ &= H(f^{k+1}(\gamma((x_0,0),t)) \cap f^{-i+1}(\gamma_1((0,y_m),s))) \\ &= d^{k+1}(\gamma_2(((x_0)',0),t)) \cap d^{-i+1}(\gamma_3((0,(y_m)'),s)) \\ &= d(d^k(\gamma_2(((x_0)',0),t)) \cap d^{-i}(\gamma_3((0,(y_m)'),s))) \\ &= d \circ H(x,y). \end{aligned}$$

Open sets having as boundaries the segments of μ-horizontal and μ-vertical curves from the foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{P}_1$ and $\mathcal{P}_2$ form a base for the topology on U and Q1$'$, respectively. Therefore H is a homeomorphism. ∎

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